*Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 4, 1996*

# TWO-LAYER GRAVITY FLOW OF A HEAVY FLUID ABOVE A CURVILINEAR BOTTOM

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UDC 532.5

The class of hydrodynamic and aerodynamic free-surface problems with internal velocity-discontinuity surfaces is of great theoretical and practical interest.

In problems of jet collisions, pneumonics, interaction of flows with different dynamic pressures, one deals with flows with internal discontinuity surfaces. In particular, the problem of internal waves in a two-layer liquid is classed among these problems.

There are numerous methods for solving problems of two-dimensional flow with different Bernoulli constants both in linear [1, 2] and nonlinear [3–9] formulations. A considerable number of publications in this line are devoted to problems of internal waves.

An effective method has been proposed [10–12] for precise nonlinear calculations of plane and axisymmetric flows of an ideal fluid with free surfaces. The present paper extends this method to the case of flow with discontinuity surfaces of the Bernoulli constant.

The method is described and tested by calculation of the problem of potential two-layer flow above a curvilinear bottom in the class of soliton-type solutions. The particular case of this problem with a rectilinear bottom has been studied by a number of authors.

The proposed method allows one to obtain a numerical solution of the problem for a curvilinear bottom of an arbitrary shape, while the most effective method of calculating such flows [9] is applicable only for the particular case of a bottom irregularity in the shape of a circular half-cylinder.

The accuracy of the method is tested by comparison of numerical and exact solutions of a certain model problem.

Let us consider a two-dimensional problem of steady two-layer potential flow of a heavy fluid over an obstruction located on a horizontal straight bottom. The parameters for the lower layer are denoted by subscript 1, and those for the upper layer, by subscript 2. The flow diagram is shown in Fig. .

Soliton-type solutions symmetric about the vertical axis $x = x_0/2$ ($x_0$ is the length of the obstruction) are sought for the free-surface and interface shapes.

Let $V_1^+$ and $V_1^-$ denote the flow velocity along the inner interface with approach to this interface from above and from below, respectively.

The condition of pressure continuity for passage through the interface, according to the Bernoulli equation, leads to the following relation for velocities on the sides of the interface:

$$\left(\frac{V_1^-}{V_{\infty 1}}\right)^2 = 1 - \frac{\rho_2 V_{\infty 2}^2}{\rho_1 V_{\infty 1}^2}\left[1 - \left(\frac{V_1^+}{V_{\infty 2}}\right)^2\right] - \frac{2}{\mathrm{Fr}_1^2}\left(1 - \frac{\rho_2}{\rho_1}\right)\left(\frac{y}{H_1} - 1\right). \tag{1}$$

Here $\rho_1$ and $\rho_2$ are the fluid densities, $\mathrm{Fr}_1 = V_{\infty 1}/\sqrt{gH_1}$ is Froude's number for the lower layer, $g$ is the acceleration of gravity, and $H_1$ is the interface ordinate at infinity.

The constant-pressure condition along the free surface is equivalent to the following law of velocity distribution along it:

$$\frac{V_2^2}{V_{\infty 2}^2} = 1 - \frac{2}{\mathrm{Fr}_2^2}\left(\frac{y}{H_2} - 1\right), \tag{2}$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 4, pp. 62–66, July–August, 1996. Original article submitted June 13, 1995.

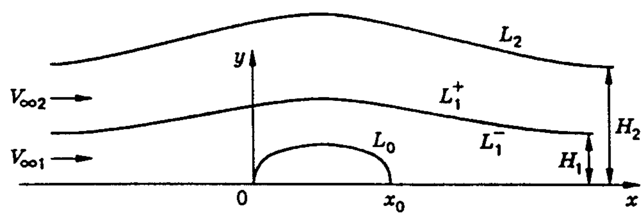


Fig. 1

where $\mathrm{Fr}_2 = V_{\infty 2}/\sqrt{gH_2}$ and $H_2$ is the free-surface ordinate at infinity.

Condition (2) is a particular case of condition (1) if the fluid density above the free-surface is assumed to be equal to zero.

The stream function of the potential flow of an ideal fluid at an arbitrary point $z = x + iy$ in the flow region can be written as

$$2\pi\Psi(z) = \int\limits_{L_0} V_0(s)R(z,\zeta)dl(s) - \int\limits_{L_1^-} V_1^-(s)R(z,\zeta)dl(s) - \int\limits_{L_1^+} V_1^+(s)R(z,\zeta)dl(s) - \int\limits_{L_2} V_2(s)R(z,\zeta)dl(s), \quad (3)$$

where $(L_0, V_0)$, $(L_1^-, V_1^-)$, $(L_1^+, V_1^+)$, and $(L_2, V_2)$ are the boundary surfaces and velocity distributions along them, the function $R(z,\zeta)$ is expressed in terms of the distances $r_1 = |z - \bar{\zeta}|$ and $r_2 = |z - \zeta|$ between the points $z$, $\bar{\zeta} = x(s) - iy(s)$, and $z$, $\zeta$ by formula $R(z,\zeta) = \ln(r_2/r_1)$.

When the point of observation tends to the boundary $z \to z_0 = x(t) + iy(t)$ in passage to the limit in Eq. (3), satisfaction of the condition of nonpenetration, which is equivalent to the condition of constant-stream-function values on each of the corresponding boundaries $L_0$, $L_1^+$, $L_1^-$, and $L_2$, leads to a system of four integral equations for the desired functions $V_0(s)$, $V_1^+(s)$, $Y_1^+(s)$, and $Y_2(s)$. These functions describe parametrically the flow-velocity distribution along the obstruction and along the outer (with respect to the bottom) interface between the liquids, and the shapes of the interface and free surface.

The condition of flow symmetry about the vertical $x = x_0/2$ is used in the solution of the problem.

Along a given boundary $L_0$ we introduce the integration variable $0 \leqslant s \leqslant 1$, which is related to the arc length by

$$l = l(1)s^{1-\alpha},$$

where $l(1)$ is the total arc length of the symmetric section and $\alpha\pi$ is the angle between the $x$ axis and the tangent to the obstruction at the stagnation point.

For $\alpha = 1/2$, it is convenient to introduce the integration variable $s$ given by the formula

$$x(s) = \frac{x_0}{2}\left(1 - \cos\frac{s\pi}{2}\right), \quad 0 \leqslant s \leqslant 1.$$

The form of the function $y(s)$ on this part of the boundary is determined by the equation of the obstruction surface $y = F(x)$. In the particular case of an obstruction in the shape of a half-ellipse with aspect ratio $\lambda$, we obtain

$$y(s) = \frac{x_0}{2\lambda}\sin\frac{s\pi}{2}.$$

Let us introduce the function $V_*$ defined by the formula

$$V_*(s) = V_0(s)\frac{dl}{ds}.$$

Taking into account the flow-velocity distribution law in the neighborhood of the stagnation point $V_0 \approx cl^{\alpha/(1-\alpha)}$, we find that for small $s$ both the function

$$V_*(s) = V_0(s)\frac{dl}{ds} \approx c_1 l^{\alpha/(1-\alpha)}s^{-\alpha}$$

and its derivatives are bounded functions.

Thus, the chosen substitution of the integration variable eliminates the singularity of the integrand in (3) at the stagnation point in integration along $L_0$.

In the numerical solution of the problem, at fairly great distances from the body upstream (at $x \leqslant X_k$) and downstream (at $x \geqslant x_0 + X_k$) the flow is considered uniform. Accordingly, the interface and free boundary on these sections extending to infinity are considered linear.

The adopted assumptions allow us to calculate analytically the integrals along semi-infinite linear parts of the boundaries $L_1^-$, $L_1^+$, and $L_2$. Let

$$x(s) = \frac{x_0}{2} s - X_k(1-s), \quad 0 < s < 1$$

on the remaining symmetric parts of the boundaries.

If points $z$ and $\zeta(s = t)$ coincide, the integrand function in (3) has a logarithmic singularity. Integrals with a logarithmic singularity are calculated using the substitution of variables $u^r = |t - s|$. The integer number $r$ is chosen on the basis of the required smoothness of the integrand function for the new variable ($r = 5$ is assumed in the calculations).

The dependence $V_*(s)$ is found as a function of the variable $u$, which is related to the integration variable $s$ by $u = \sin(s\pi/2)$. This substitution ensures that the function $V_*(u)$ is close to linear for a circular half-cylinder. For an obstruction of this shape this function would be linear in the case of one-layer flow of infinite depth.

Satisfaction of the condition of nonpenetration along the boundary streamlines and the interface for a stream function in the form of (3) yields a system of integral equations for determining functions of flow-velocity distribution along the specified boundary (bottom profile) and interface and also for determining functions that describe the shape of the unknown boundaries: the Bernoulli constant discontinuity line and the free boundary.

In the problem with a "cover," when the external-boundary shape is given, the velocity distribution along the external boundary is included in the number of the desired functions.

The thus obtained system of four integral equations is solved by the spline collocation method. The unknown functions are approximated by cubic splines. A system of node points is introduced along each boundary. The conditions of satisfaction of integral equations at the interpolation nodes leads to a closed system of transcendental equations with respect to the values of unknown functions at these points. It is solved numerically by extending the Steffensen method for finding roots of a one-variable function to the $n$-dimensional case [13]. Such an algorithm has been previously applied to calculations of a number of free-surface problems [10–12, 14].

The accuracy of the numerical method was checked by calculating the following model (test) problem using this method. Let the obstruction have the shape of a half-cylinder with unit radius. The "free" boundary will be determined using, instead of boundary condition (2), the following velocity distribution law:

$$\frac{V_2^2}{V_{\infty 2}^2} = \left(1 - \frac{1}{\rho^2} + \frac{2y^2}{\rho^4}\right)^2 + \frac{4y^2(1-x)^2}{\rho^8}, \quad \rho^2 = (1-x)^2 + y^2.$$

The internal "interface" will be found using boundary condition (1) with $\rho_2 = \rho_1$ and $V_{\infty 2} = V_{\infty 1}$.

An analytical solution of this auxiliary problem is readily obtained from the known solution of the problem of noncirculatory potential flow over a circular cylinder. In particular, the shapes of the "free" boundary and "interface" are defined by the relations

$$H_i - y(1 - \rho^{-2}) = 0, \quad i = 1, 2.$$

The accuracy of the method is fairly high even for a small number of node points. Let us illustrate this by comparison of numerical and exact solutions for $H_1 = 1$ and $H_2 = 2$. In this case, the exact solution yields $Y_{1\max} = 1.618$ and $Y_{2\max} = 2.414$, while the numerical solution with four nodes for each of the symmetric parts of the boundary gives $Y_{1\max} = 1.618$ and $Y_{2\max} = 2.415$.

TABLE 1

| $H_2/H_1$ | $a/H_1$ | $\rho_2/\rho_1$ | $\mathrm{Fr}_1$ | $\mathrm{Fr}_1$ [15] |
|---|---|---|---|---|
| 3.216 | 0.459 | 0.8 | 0.418 | 0.419 |
| 3.216 | 0.2 | 0.05 | 1.051 | 1.052 |
| 3.15 | 0.3 | 0.8 | 0.406 | 0.408 |
| 2 | 0.2 | 0.3 | 0.758 | 0.761 |
| 1.5 | 0.2 | 0.05 | 0.972 | 0.975 |

The results of the numerical solution for the straight cover problem of two-layer flow above a rectilinear bottom are listed in Table 1. They are compared with the results of the approximate solution of [15] which was obtained in a second approximation of shallow water theory.

In the calculation of this problem the wave amplitude $a$ was specified, and Froude's number $\mathrm{Fr}_1$ was the desired parameter.

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